Chapter 7 A Definite Integral Whose Indefinite Form Cannot be Done

As you know by now, evaluating integrals is harder than computing derivatives. Another complication is that there are functions $f(x)$, written in terms of functions you know, whose integrals *cannot* be written in terms of functions you know. To complicate things further, even if $\int f(x)\,dx$ is such a function, it may be possible to evaluate $\int_a^b f(x)\,dx$ for some values of a and b. Of course there are trivial cases such as $a = b$ when the integral is zero. What about nontrivial examples?

One of the most popular ones to show students is $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. This is done using multiple integrals, so it can't be done in this course. Here is an example that can be done:

$$\int_0^{\pi/2} \ln(\sin x)\;dx. \tag{1}$$

You might like to try doing $\int \ln(\sin x)\,dx$. You'll find that you cannot.

The integral in (1) is improper because the function is infinite at $x = 0$. In this paragraph, we show that it converges. Since $\sin x \geq 2x/\pi$ for $0 \leq x \leq \pi/2$, we have $\ln(\sin x) \geq \ln(2x/\pi)$. Thus

$$0 \leq -\ln(\sin x) \leq -\ln x + \ln(2/\pi).$$

Since $\int \ln x\,dx = x\ln x - 1$, you should be able to show that $\int_0^{\pi/2} \ln(\sin x)\,dx$ converges.

We now evaluate the integral using $\sin x = 2\sin(x/2)\cos(x/2)$ and substitution. First

$$\begin{aligned}\int_0^{\pi/2} \ln(\sin x)\;dx &= \int_0^{\pi/2} \ln\Big(2\sin(x/2)\cos(x/2)\Big)\;dx \\ &= \int_0^{\pi/2} \ln 2\;dx + \int_0^{\pi/2} \ln(\sin(x/2))\;dx + \int_0^{\pi/2} \ln(\cos(x/2)\;dx.\end{aligned}$$

Now let $x/2 = u$ in the last two integrals to obtain

$$\int_0^{\pi/2} \ln(\sin x)\;dx = (\pi\ln 2)/2 + 2\int_0^{\pi/4} \ln(\sin u)\;du + 2\int_0^{\pi/4} \ln(\cos u)\;du.$$

Now let $u = \pi/2 - t$ in the last integral to obtain

$$\int_0^{\pi/4} \ln(\cos u)\;du = -\int_{\pi/2}^{\pi/4} \ln(\sin t)\;dt = \int_{\pi/4}^{\pi/2} \ln(\sin t)\;dt,$$

since $\cos(A - B) = \cos A\cos B + \sin A\sin B$. Putting all this together we have

$$\begin{aligned}\int_0^{\pi/2} \ln(\sin x)\;dx &= (\pi\ln 2)/2 + 2\int_0^{\pi/4} \ln(\sin u)\;du + 2\int_{\pi/4}^{\pi/2} \ln(\sin t)\;du \\ &= (\pi\ln 2)/2 + 2\int_0^{\pi/2} \ln(\sin x)\;dx.\end{aligned}$$

Hence the value of the integral in (1) is $-(\pi\ln 2)/2$.